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Stress dependent behaviour of a particulate material

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This paper considers the non-linear stress–strain characteristics of a particulate material using the classical elasticity concepts of modulus of elasticity and Poisson's ratio and shows that these terms are stress-dependent. In addition to the first loading of the material a number of unloading/reloading cycles are carried out under conditions representative of practical loading cases. Finally, recommendations are made regarding the analysis of problems involving particulate materials subjected to three-dimensional stress and strain.

1. Introduction

A particulate material is an assemblage of particles of various shapes and sizes, with or without the presence of water, of which the natural stability depends on the interlocking of the particles and the friction between them. A knowledge of the behaviour of such a material is of interest to soil mechanicians and materials scientists involved in the study of such problems as response to loading, soil stability and packing and pressing processes.

From general observation of the stress–strain behaviour of particulate materials, it is recognised that their response to loading is complex, and involves so many redundancies, that a solution based on a study of the equilibrium of individual particles of the assemblage is virtually impossible. Early mathematical models used to represent the behaviour of particulate materials tended to favour existing concepts from the theory of elasticity, although it was recognised that the modulus of elasticity and Poisson's ratio were not constant but stress-dependent. A more refined method of approach is to study the load–deflection curves, and hence the stress–strain curves, of the assemblage, under controlled conditions. The work is similar to that done on such materials as rock, concrete, or coal, which have been tested to destruction under vertical stress. There is, however, the basic difference that an assemblage of particles requires lateral support, and hence the experiments consist of a series of tests in which the material is allowed to yield under constant lateral stress, and the lateral strains measured. An approach of this nature [1] has attempted to describe the non-linear stress-dependent inelastic stress–strain behaviour by an incremental representation with a stress-dependent tangent modulus, evaluated for the average stress conditions during the increment and a constant value for Poisson's ratio. These values were obtained from standard triaxial test results in which the intermediate principal stress is equal to the minor principal stress. The approach also took account of sample behaviour during unloading/reloading cycles which, consistent with other writers [2, 3], was shown to have a near-linear reponse. Whether it is valid to use such results for the analysis of plane strain or more involved problems is a point worthy of further investigation.

An approach for describing the behaviour of granular media when subjected to a three-dimensional stress system has been formulated by the present authors [4]; this relates incremental stress and strain by a number of stress dependent coefficients. Besides being applicable for unloading/reloading cycles, the approach offers the dual advantage of including the intermediate principal stress in both the elasto-plastic behaviour and rupture condition, and also permits claculation of the stress-dependent modulii of elasticity and Poisson's ratios, if required. It should be emphasised, however, that this is incidental and that the original formulation was derived independently of classical elasticity. The work has already been used for a number of problems in soil mechanics [5–7]. and it is proposed to use this approach to investigate the variation of the stress-dependent elastic constants for different states of loading; in particular, their compairson between loading in which the intermediate and minor principal stresses are equal, and that of plane strain.

Finally, a new stress–strain theory for a granular material has recently been formulated [8] which incorporates both elastic and plastic deformations and expresses the failure criteria in terms of stress invariants. The theory appears to offer an attractive method for predicting the behaviour of a granular material but, as presented, is unlikely to be generally applicable [9]. However, it is considered that an approach of this nature may form the basis of future research into this area of study.

2. Non-linear characteristics of granular media

It is recognised that a number of factors govern the stress–strain behaviour of particulate materials. These include the characteristics of the particles, water content and drainage conditions, type and duration of loading and stress history. The latter items have attracted considerable interest because sample behaviour is influenced by the anisotropy which is induced by the loading system.

As the basis of an incremental non-linear description of granular media behaviour has already been published in some detail [4] it is considered only necessary to repeat the salient features of the work for the present purposes. The relationships presented describe the loading, unloading and reloading of a dry granular material and are incremental stress–strain equations of the following type:

$$\begin{aligned}
\delta\epsilon_x &= \frac{\delta p_x}{\cdot K\left(p_x - \frac{1}{K_r}\sqrt{(p_y p_z)}\right)} - \left(2 - \frac{p_x}{p_z}\right)\left[\frac{m}{\left(\frac{p_x}{K_r} - p_y\right)} - \frac{n}{\frac{p_x}{K_r}}\right]\delta p_y - \left(2 - \frac{p_x}{p_y}\right)\left[\frac{m}{\left(\frac{p_x}{K_r} - p_z\right)} - \frac{n}{\frac{p_x}{K_r}}\right]\delta p_z \\
\delta\epsilon_y &= -\frac{p_z}{p_y}\left[\frac{m}{\left(\frac{p_z}{K_r} - p_x\right)} - \frac{n}{\frac{p_z}{K_r}}\right]\delta p_x + \frac{\delta p_y}{K\left(p_y - \frac{1}{K_r}\sqrt{(p_x p_z)}\right)} - \frac{p_x}{p_y}\left[\frac{m}{\left(\frac{p_x}{K_r} - p_z\right)} - \frac{n}{\frac{p_x}{K_r}}\right]\delta p_z \qquad (1) \\
\delta\epsilon_z &= -\left(2 - \frac{p_z}{p_y}\right)\left[\frac{m}{\left(\frac{p_z}{K_r} - p_x\right)} - \frac{n}{\frac{p_z}{K_r}}\right]\delta p_x - \frac{p_x}{p_z}\left[\frac{m}{\left(\frac{p_x}{K_r} - p_y\right)} - \frac{n}{\frac{p_x}{K_r}}\right]\delta p_y + \frac{\delta p_z}{K\left(p_z - \frac{1}{K_r}\sqrt{p_x p_y}\right)}
\end{aligned}$$

for which

$$p_x \leqslant p_z \leqslant p_y,$$

where p_x, p_y, p_z are the stresses and ϵ_x, ϵ_y, ϵ_z the strains in the respective co-ordinate directions, δp_x, $\delta\epsilon_x$ etc. small increments of the quantities concerned, K_r the Rankine ratio, and K, m and n constants found experimentally.

If we represent the behaviour of the material by the generalized Hooke's law, in which the incremental strains for each load increment are related to the incremental stress changes, the following relationships are applicable:

$$\begin{aligned}
\delta\epsilon_x &= \frac{\delta p_x}{E_x} - \frac{\nu_{xy}}{E_y}\delta p_y - \frac{\nu_{xz}}{E_z}\delta p_z \\
\delta\epsilon_y &= -\frac{\nu_{yx}}{E_x}\delta p_n + \frac{\delta p_y}{E_y} - \frac{\nu_{yz}}{E_z}\delta p_z \qquad (2) \\
\delta\epsilon_z &= -\frac{\nu_{zx}}{E_x}\delta p_x - \frac{\nu_{zx}}{E_y}\delta p_y + \frac{\delta p_z}{E_z}
\end{aligned}$$

where E_x, E_y, E_z are stress-dependent modulii of elasticity and ν_{xy} etc. Poisson's ratios respectively.

Comparison of the corresponding terms of Equations 1 and 2 permits calculation of the stress dependent modulii of elasticity and Poisson's ratio.

3. Results

The following calculations were carried out in order to investigate the variation of the modulii of elasticity and Poisson's ratios under different loading states. These are representative of a number of practical loading conditions used in the assessment of test data for engineering soils and, in addition, simulate the type of loading variations

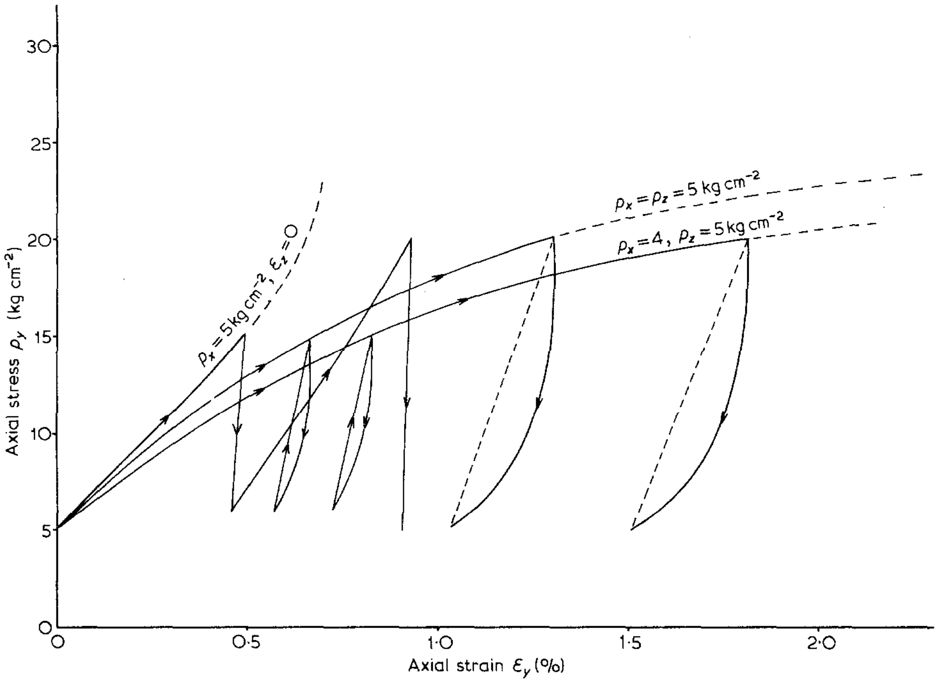

Figure 1 Stress–strain behaviour of granular sample.

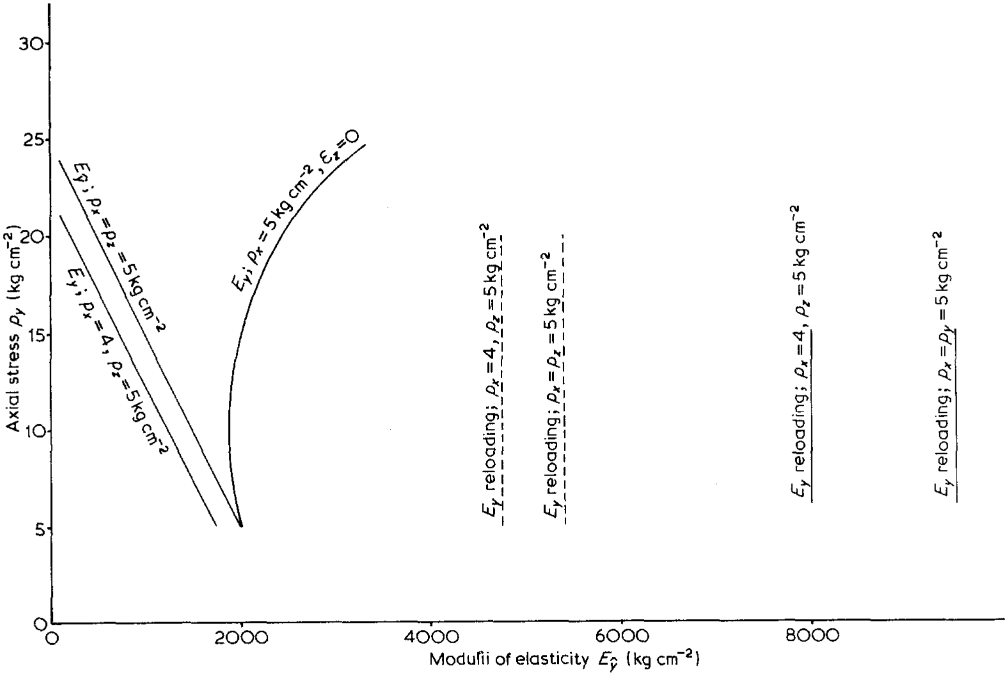

Figure 2 Variation of modulii of elasticity.

that are likely to occur in a bed of material beneath structural foundations.

(i) Standard triaxial type loading with $p_x = p_z = p_h = 5.0$ kg cm^{-2};

loading p_y = 5.0 to 15.0 kg cm^{-2},
unloading p_y = 15.0 to 6.0 kg cm^{-2},
reloading p_y = 6.0 to 20.0 kg cm^{-2},
unloading p_y = 20.0 to 5.0 kg cm^{-2}.

(ii) Axial loading under plane strain condition with $p_x = 5.0$ kg cm^{-2} and $\epsilon_z = 0$; Loading/unloading, etc. as in case (i).

(iii) Axial loading with unequal confining

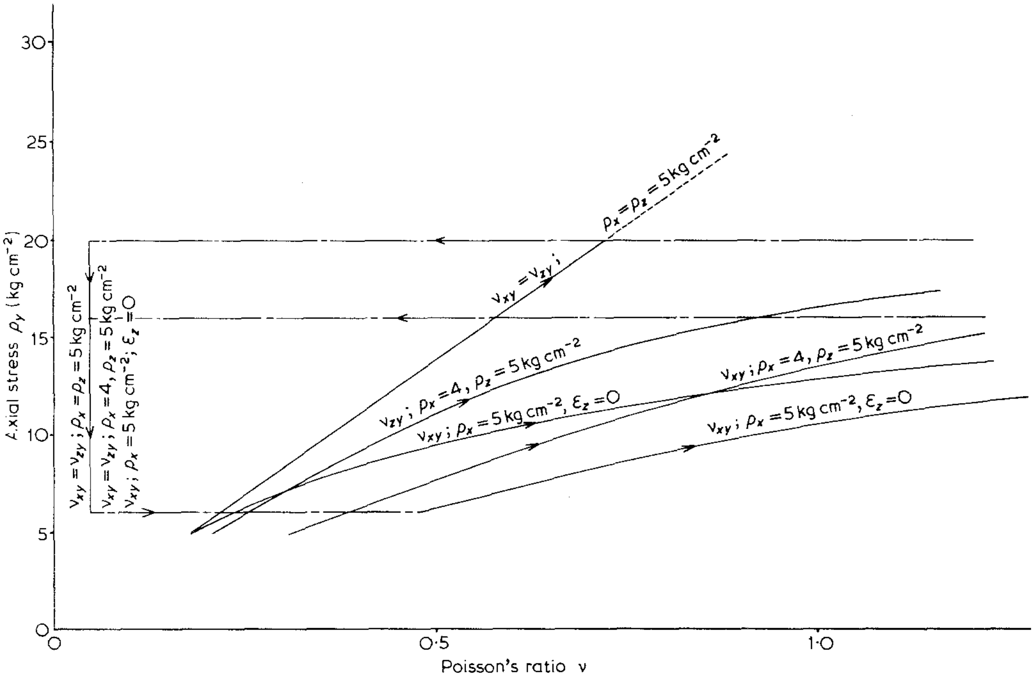

Figure 3 Variation of Poisson's ratio.

stresses with $p_x = 4.0$ kg cm^{-2} and $p_z = 5.0$ kg cm^{-2}; Loading/unloading, etc. as in case (i). The results of these calculations are shown in Figs. 1 to 3.

The above approach can be applied to more general loading states to which sand samples may be subjected, and this can form the basis of the assessment of bed behaviour under a variety of loading conditions. An example of this approach is described in [7] and, for more involved problems, a sophisticated non-linear finite element analysis can be implemented using the proposed formulation. As the bed is loaded, as incremental analysis can establish the stress field associated with each element of the bed, and hence the stress-dependent behaviour of each element. An accurate description of the stress-dependent behaviour during loading and unloading is important as the assumption of a simplified relationship can lead to gross inaccuracies. A comprehensive incremental analysis of all the elements of the bed establishes stress paths compatible with the local deformations and imposed boundary conditions leading to realistic predictions of, for example, load–displacement behaviour. The calculation of such information, which for civil engineers may involve the settlement of a foundation or displacement of a ground anchor and for materials scientists the packing of materials, is of importance for optimization in engineering design.

4. Discussion

The axial stress–strain behaviour for the three investigations are shown in Fig. 1 where both primary loading and unloading/reloading cycles are drawn. The dashed portion of the graphs represents loading continuation outside the specified loading range. For samples subjected to constant lateral stresses under triaxial compression, increasing axial stress results in decreasing axial stiffness until failure takes place [10]. At this stage the stress-dependent modulus of elasticity E_y has decreased to zero, and this is consistent with the rupture condition of the material. Of particular interest is the unloading/reloading cycle where it can be seen that the unloading/reloading modulus is not a function of the confining stress only, as has been shown by others [1], but is dependent on the stress levels attained during the cycling process.

During loading under plane strain conditions, variation of the confining stress in the z-direction takes place in order to maintain the required strain condition. This affects the primary loading relationship, and the behaviour after an unloading/reloading cycle is such that the subsequent reloading relationship is not coincident with that of first loading as is found for constant confining stresses.

Although the overall lateral strain/axial strain ratio during the testing of a triaxial specimen may eventually be of the order of 0.9 or 1.0 thus indicating dilation, this effect is made up of several components which, when isolated, indicate a rather more complex behaviour than is initially apparent. It is evident, as stated previously [11], that the assumption of a common, though stress-dependent, modulus of elasticity in all the stress–strain equations is incorrect. Examination of Fig. 2 shows that, if lateral stresses are constant, E_y reduces during loading, and this is consistent with the results of other writers. However, under more general loading conditions, the modulii of elasticity do not behave in a similar fashion and must be considered individual to each axis as is typical of anisotropic media. This is clearly demonstrated in the case of plane strain where E_y exhibits quite different behaviour.

The modulus during the unloading portion of the unloading/reloading cycle varies depending on the degree of unloading and, for clarity, constant reloading modulii only are shown in Fig. 2.

The stress dependent Poisson's ratios, see Fig. 3 in which a number of these are shown, vary considerably and it is clear that an assumption of constancy over a range of stressing would be an extreme simplification. It has generally been considered that the value of Poisson's ratio usually has a relatively small effect upon engineering predictions, particularly during the early stages of loading. However, if a correct evaluation of the modulus of elasticity is made on the basis of an individual stress-dependent modulus in each stress–strain equation, it is clear that Poisson's ratio must, in some cases, be large for a theoretical formulation to predict the dilatational behaviour of the granular specimen. This commonly observed behaviour during the first loading of granular media is due to the influence of the lateral movement of the particles on the void ratio, and can be followed, when the specimen is subjected to unloading/reloading cycles, by compaction.

5. Conclusions

The behaviour of the specimen is complex and the purpose of this investigation has been to highlight the deficiencies in conventional approaches, and offer tractable alternatives to these procedures. A summary of the main points is as follows:

(i) The assumption of a common, though stress-dependent, modulus of elasticity in the stress–strain relationship is, in general, invalid. Furthermore, a constant Poisson's ratio, even for the early stages of loading, could lead to incorrect results in subsequent analysis.

(ii) The results from standard triaxial testing for use in plane strain analysis is questionable. It is recognized that the plane strain condition is a common occurrence in problems involving the analysis of engineering soils and, for work of the nature, a plane strain form of test should be used in preference to the standard test [12].

(iii) It is considered that the best approach to problems of this nature would be to dispense with the application of a modified classical elasticity approach and to recognize that particulate media deserve a quite independent appraisal. For this purpose the work of the present authors [4] is recommended, particularly for problems involving three-dimensional stress and strain.

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